Exercise 5H: Solutions

1 Label three holes with the colours red, blue and green.

$$\boxed{\text{R}}\ \boxed{\text{B}}\ \boxed{\text{G}}$$

Select four socks and place each sock in the hole whose label corresponds to the colour of the sock. As there are four socks and three holes, the Pigeonhole Principle guarantees that some hole contains at least two socks. This is the required pair. Clearly, selecting three socks is not sufficient as you might pick one sock of each colour.

2 Create 26 holes, each labelled with a letter from A to Z.

$$\boxed{\text{A}}\ \boxed{\text{B}} \cdots \boxed{\text{Z}}$$

Put each word in the hole whose label corresponds to its first letter. There are 27 words to place in 26 holes, so there is some hole that contains at least 2 words. Therefore, there are at least two words that begin with the same letter.

3 Create 4 holes with labels $0, 1, 2$ and 3.

$$\boxed{0}\ \boxed{1}\ \boxed{2}\ \boxed{3}$$

Put each natural number in the hole whose label corresponds to the remainder when it is divided by 4. There are 5 numbers to place in 4 holes, so there is some hole that contains at least 2 numbers. Therefore, there is at least 2 numbers that leave the same remainder when divided by 4.

4 **a** As there are only 2 colours, if 3 cards are dealt, at least two will be the same colour.

b As there are only 4 suits, if 5 cards are dealt, there is some suit that occurs at least twice.

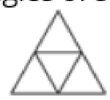
c As there are 13 kinds, if 14 cards are dealt, there is at least one kind that occurs at least twice.

5 Divide the number line into ten intervals:

$$[0, 0.1), [0.1, 0.2), \ldots [0.9, 1].$$

Eleven points are located in 10 intervals, so some interval contains at least two numbers. The distance between any two number in this interval is no more than 0.1 units.

6 Divide the equilateral triangle into 4 equilateral triangles of side length 1 unit, as shown:



There are 5 points located in 4 triangles, so some triangle contains at least 2 points. The distance between any two points in this triangle is no more than 1 unit.

7 Divide the rectangle up into square of size 2 metres by 2 metres. There are 13 points located in $3 \times 4 = 12$ squares, so some square contains at least two points. The distance between any two of these points can not exceed length of the square's diagonal, $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

8 **a** For two-digit numbers, there are 18 possible digital sums: $1, 2, \ldots, 18$. So if there are 19 numbers then there there is some digital sum that occurs at least twice.

b For three-digit numbers, there are 27 possible digital sums: $1, 2, \ldots, 27$. Since $82 = 3 \times 27 + 1$, by the Generalised Pigeonhole Principle, there is some digital sum that occurs at least 4 times.

9 Label four holes with each of four possible remainders when a number is divided by 4, namely $0, 1, 2$ and 3. Each of the 13 numbers belongs to one of 4 holes. Since $13 = 3 \times 4 + 1$, there is some hole that contains at least 4 numbers.

10 There are ${}^8C_2 = 28$ ways that 2 teams can be chosen to compete from 8 choices. There are 29 games of football, so there is some pair of teams that play each other at least twice.

11 Label 25 holes as shown below

$$\boxed{1 \text{ or } 49}\ \boxed{2 \text{ or } 48} \cdots \boxed{24 \text{ or } 26}\ \boxed{50}$$

If there are 26 numbers, then there is some hole that contains at least 2 numbers. As these two numbers are distinct, their sum must be 50. Clearly, this is the smallest number of students required, since 25 students could select the numbers $1, 2, 3, \ldots, 25$, no two of which add to 50.

12 Label the chairs with numbers from 1 to 14. There are 14 groups of three consecutive chairs:

$\{1,2,3\}, \{2,3,4\}, \ldots \{13,14,1\}, \{14,1,2\}$.

Each of the 10 people will belong to 3 of these groups, so there are 30 people to be allocated to 14 groups. Since

$$30 > 2 \times 14 + 1,$$

the Generalised Pigeonhole Principle guarantees that some groups must contain 3 people.

13 Pick any one of the 4 points and draw a diameter through that point. This splits the circle into two half circles. Considering the remaining 3 points, the pigeonhole principle says that one of the half circles must contain at least 2 of those 3 points. Together with the initial point chosen, that half circle contains at least 3 points.

14 Using two distinct integers chosen from $\{1, 2, \ldots, 98, 99\}$, there are 195 different sums possible: $3, 4, \ldots, 197$. There are 35 different players and ${}^{35}C_2 = 595$ ways to pair these players up. Since

$$595 = 3 \times 195 + 10 > 3 \times 195 + 1,$$

the Generalised Pigeonhole Principle guarantees that at least four pairs will have the same sum.

15 Label the chairs with numbers from 1 to 12 and then create 6 pairs of opposite seats:

$\{1,7\}, \{2,8\}, \{3,9\}, \{4,8\}, \{5,9\}, \{6,12\}$.

Each of the 7 boys belongs to one of 6 pairs, so some pair contains two boys.

16 Create n holes with labels

$$0, 1, 2, \ldots, n-1.$$

Place each of the n guests in the hole corresponding to the number of hands that they shake. Note that either the first or last hole must be empty since if some guest shakes hands with 0 people then no guest shakes hands with all people. Likewise, if some guest shakes hands with all people, then no guest shakes hands with 0 people. This leaves $n-1$ holes in which n guests are located. The Pigeonhole Principle guarantees that some hole has at least two guests.